Published by Institute of Physics Publishing for SISSA/ISAS
Received: *September 16, 2003*
Revised: *January 16, 2004*
Accepted: *January 21, 2004*

Rolling G_2 moduli

André Lukas and Stephen Morris

Department of Physics and Astronomy, University of Sussex
Falmer, Brighton BN1 9QH, U.K.
E-mail: a.lukas@sussex.ac.uk, s.morris@sussex.ac.uk

Abstract: We study the time evolution of freely rolling moduli in the context of M-theory on a G_2 manifold. This free evolution approximates the correct dynamics of the system at sufficiently large values of the moduli when effects from non-perturbative potentials and flux are negligible. Moduli fall into two classes, namely bulk moduli and blow-up moduli. We obtain a number of non-trivial solutions for the time-evolution of these moduli. As a generic feature, we find the blow-up moduli always expand asymptotically at early and late time.

Keywords: M-Theory, Compactification and String Models.

http://jhep.sissa.it/archive/papers/jhep012004045/jhep012004045.pdf

JHEP01(2004)045

Contents

1. Introduction

Seven-dimensional manifolds with holonomy G_2 provide the general framework for relating M-theory to four-dimensional models with $N = 1$ supersymmetry. Moreover, it has been shown that for certain singular limits of such manifolds four-dimensional models with phenomenologically interesting properties can be obtained [1]–[16].

Given these promising features it seems worthwhile to have a closer look at the cosmology of such G_2 compactifications of M-theory. More specifically, in this paper, we will be interested in the time-evolution of G_2 moduli. This problem has not previously been analysed since the four-dimensional effective theories, specifically the kinetic terms, for these fields were unknown. However, in a recent paper [17], the four-dimensional moduli Kähler potential for M-theory on a G_2 manifold has been computed explicitly and our analysis will be based on these results.

The G_2 manifold considered in ref. [17] is based on the compact examples due to Joyce [18]–[20], constructed by blowing-up a seven-dimensional G_2 orbifold. The moduli for such a G_2 manifold split into two groups, namely the bulk moduli (related to the radii of the underlying seven-torus) and the blow-up moduli (related to the volume of the blow-ups). The Kähler potential of ref. [17] was obtained in the usual large-radius supergravity approximation and for blow-up volumes small compared to the total volume of the space. In the limit of a vanishing blow-up modulus the space develops a singularity (which corresponds to a singularity of the underlying orbifold) leading to new fields in the low-energy theory. It is singular limits such as this which are essential for a phenomenologically interesting low-energy theory.

In this paper, we will focus on the simple case of free moduli evolution based on the Kähler potential of ref. [17]. That is we will neglect the effect of possible non-perturbative potentials and flux. Hence our results constitute a good approximation at large moduli values where effects of a non-trivial moduli (super)-potential can be neglected. Particular consideration is given to the evolution of the blow-up moduli, particularly as to whether they may contract, thereby approaching one of the phenomenologically interesting singular limits.

Let us summarise our main results. Starting from the given Kähler potential and consistently truncating off the axions in the chiral moduli multiplets, we show that, after a suitable reparametrisation of fields, the sigma-model metric becomes independent of the blow-up moduli. As a result, these fields can be integrated out and the problem reduces to one of studying the evolution of the bulk moduli in an effective potential. Based on this approach, we show that a number of explicit analytic solutions can be obtained using methods from Toda theory. For all these solutions, we find the relative volume of the blow-up moduli increases asymptotically at early and late times. As a consequence, the small blow-up approximation underlying the Kähler potential breaks down after a finite proper time. This feature can also be intuitively understood from the properties of the effective bulk-moduli potential and can, therefore, be viewed as generic. Hence, once moduli are large it appears to be difficult to evolve towards a singular state of the internal space. This suggests that a successful cosmological evolution leading to phenomenologically interesting singular G_2 spaces should stabilise moduli at small values from the outset.

2. Kähler potential for G_2 moduli

In ref. [17] the moduli Kähler potential has been calculated for a specific manifold with G_2 holonomy constructed from a G_2 orbifold by blowing up the orbifold singularities. The chiral moduli multiplets for this manifold split into two categories, namely the bulk moduli T^A, where $A, B, C, \dots = 1, \dots, 7$, which encode the radii of the seven-torus T^7 underlying the orbifold, and the blow-up moduli U^i, which measure the volume of the blow-ups. The general structure of this Kähler potential is

$$K = -\sum_{A=1}^{7} \ln\left(T^A + \bar{T}^A\right) + \sum_i F_i(T^A, \bar{T}^A)\left(U^i + \bar{U}^i\right)^2 + c\,, \tag{2.1}$$

where the functions F_i are given by

$$F_i = \frac{8}{(T^{A_i} + \bar{T}^{A_i})(T^{B_i} + \bar{T}^{B_i})} \tag{2.2}$$

and c is a constant. Here, A_i and B_i specify the two particular bulk moduli by which the blow-up modulus i is divided in the above Kähler potential. The values for these indices reflect the underlying structure of the orbifold and they can be conveniently encoded in constant seven-dimensional vectors p_i by writing

$$F_i = 2e^{-\mathbf{p}\cdot\boldsymbol{\beta}} \tag{2.3}$$

where the new fields $\boldsymbol{\beta} = (\beta_A)$ are defined by[1]

$$\mathrm{Re}(T^A) = e^{\beta_A}\,. \tag{2.4}$$

[1]Here and in the following bold-face symbols denote seven-dimensional vectors.

$\mathbf{p}_{(\tau,a)}$	$a=1$	$a=2$	$a=3$
$\tau=\alpha$	$(1,0,0,0,0,1,0)$	$(0,1,0,0,1,0,0)$	$(0,0,1,1,0,0,0)$
$\tau=\beta$	$(1,0,0,0,0,0,1)$	$(0,0,1,0,1,0,0)$	$(0,1,0,1,0,0,0)$
$\tau=\gamma$	$(1,0,0,1,0,0,0)$	$(0,0,1,0,0,1,0)$	$(0,1,0,0,0,0,1)$

Table 1: Values of vectors $\mathbf{p}_{(\tau,a)}$ which define the moduli Kähler potential for the G_2 manifold based on $T^7/\mathbb{Z}_2^3$.

We expect this form of the Kähler potential to apply to a wide class of G_2 manifolds constructed by blowing up orbifolds with the specifics of each example encoded in the vectors $\mathbf{p}_i$.

In the following we will, for concreteness, focus on the particular example of ref. [17], which is based on the orbifold $T^7/\mathbb{Z}_2^3$. The blow-ups for this particular manifold can be labeled by a triple $(i) = (\tau, n, a)$ of indices where $\tau = \alpha, \beta, \gamma$ indicates the type, that is, under which of the three $\mathbb{Z}_2$ orbifolding symmetries the associated fixed point remains invariant, $n = 1, 2, 3, 4$ labels the fixed points of equal type and $a = 1, 2, 3$ is an index associated to each fixed point that describes the orientation of the blow-up relative to the bulk. Hence, there are 12 fixed points, labeled by (τ, n), each with three associated blow-up moduli and, consequently, 36 blow-up moduli in total. For simplicity of notation, we will use the single index i to label the blow-ups whenever possible and only split into the triple (τ, n, a) when required. It turns out that the vectors $\mathbf{p}_i$ for this case only depend on the type τ and the orientation a. The resulting nine vectors are given in table 1.

It will be useful for the subsequent discussion to have an interpretation for the moduli and the various parts of the Kähler potential (2.1) in terms of the geometry of the underlying G_2 manifold. To this end, one notes that the moduli Kähler potential for M-theory on a G_2 manifold is generally related to the volume $\mathcal{V}$ of this manifold by [10]

$$K = -3\ln\left(\frac{\mathcal{V}}{2\pi^2}\right). \tag{2.5}$$

The first, T^A–dependent, part of the Kähler potential (2.1) then corresponds to the volume $\mathcal{V}_0$ of the orbifold, while the second, U^i–dependent, part measures the reduction of this volume due to the blow-ups. For the orbifold volume $\mathcal{V}_0$ one easily finds from eqs. (2.5) and (2.4) that

$$\mathcal{V}_0 = \frac{1}{8}\exp\left(\frac{1}{3}\sum_A \beta_A\right), \tag{2.6}$$

where we have used the value $c = 6\ln(8\pi) + \ln(2)$ found in ref. [17]. From a similar calculation one finds that the fraction $\epsilon_{(\tau,n)}$ by which the blow-up (τ, n) reduces the orbifold volume V_0 is given by

$$\epsilon_{(\tau,n)} = \frac{8}{3}\sum_{a=1}^{3} e^{-\mathbf{p}_{(\tau,a)}\cdot\boldsymbol{\beta}}\, u^2_{(\tau,n,a)}, \tag{2.7}$$

where we have introduced the real parts

$$\mathrm{Re}(U^i) = u_i \tag{2.8}$$

of the blow-up moduli.

JHEP01(2004)045

The Kähler potential (2.1) relies on two approximations so we should discuss its range of validity in moduli space. First, we require all moduli to be larger than one, that is

$$T^A \gg 1\,, \qquad U^i \gg 1\,, \tag{2.9}$$

for the supergravity approximation underlying the calculation of K to be valid. Second, the Kähler potential has been calculated to leading (quadratic) order in U^i/T^A and terms of order four in these ratios have been neglected. Consequently, application of the Kähler potential (2.1) should be confined to the region of moduli space where the ratios U^i/T^A are smaller than one, or more precisely, where

$$\epsilon_{(\tau,n)} \ll 1\,. \tag{2.10}$$

From the above interpretation of $\epsilon_{(\tau,n)}$ this means that the volume taken away by the blow-ups should be small compared to the volume of the orbifold. In subsequent calculations, we will consistently apply this approximation in that we neglect higher order terms in $\epsilon_{(\tau,n)}$.

3. A simple form of the lagrangian

The Kähler potential described in the previous section depends on 43 chiral superfields. We, therefore, have 43 real moduli fields from the real parts of theses superfields, which are associated to the geometry of the G_2 manifold, plus 43 axions from the imaginary parts. It is clear that the axions can be consistently set to constants and for simplicity this is what we will do in the following. We are, hence, left with the real scalar fields β_A and u_i.

However, their kinetic terms, as computed from eq. (2.1), are still fairly complicated. A substantial simplification can be achieved by introducing a new set of fields ϕ_A and z_i defined by the Ansatz

$$\beta_A = \phi_A + \sum_i c_{iA} e^{\mathbf{q}_i\cdot\boldsymbol{\phi}} z_i^2 \tag{3.1}$$

$$u_i = e^{\mathbf{s}_i\cdot\boldsymbol{\phi}} z_i\,. \tag{3.2}$$

Here, $\mathbf{c}_i = (c_{iA})$, $\mathbf{q}_i = (q_{Ai})$ and $\mathbf{s}_i = (s_{Ai})$ are constants to be determined shortly. Note that the sum over i in (3.1) singles out seven linear combinations of the fields z_i^2. The kinetic terms for β_A and u_i can now be rewritten using this field reparametrisation. For the particular choice of constants

$$\mathbf{c}_i = -4\mu_i\mathbf{p}_i \tag{3.3}$$

$$\mathbf{s}_i = \frac{1}{2}(\mu_i+1)\mathbf{p}_i \tag{3.4}$$

$$\mathbf{q}_i = \mu_i\mathbf{p}_i\,, \tag{3.5}$$

where $\mu_i \in \{-1,+1\}$ are arbitrary signs, we find that the lagrangian simplifies to

$$L = \sqrt{-g}\left(R + \frac{1}{2}\sum_A \partial_\mu\phi_A\partial^\mu\phi_A + 8\sum_i e^{-\mathbf{P}_i\cdot\boldsymbol{\phi}}\partial_\mu z_i\partial^\mu z_i\right) \tag{3.6}$$

where

$$\mathbf{P}_i = -\mu_i \mathbf{p}_i \,. \tag{3.7}$$

The obvious advantage of this form is that the rescaled blow-up moduli z_i (unlike their counterparts u_i) only appear through their derivatives. This implies that their equations of motion can be immediately integrated once, leading to a set of first integrals. We will use this property in the following section when we discuss the cosmological evolution based on the lagrangian (3.6). Since we have neglected higher order terms in $\epsilon_{(\tau,n)}$ in its derivation, it is equivalent to the one directly obtained from the Kähler potential (2.1) only provided fields are in the region of moduli space defined by eq. (2.10). Also note that this equivalence holds for an arbitrary choice of the signs μ_i.

4. Cosmological evolution equations

To study the cosmological evolution based on the lagrangian (3.6) we start with a metric

$$ds^2 = -e^{2\nu(\tau)} d\tau^2 + e^{2\alpha(\tau)} d\mathbf{x}^2 \tag{4.1}$$

of Friedmann-Robertson-Walker type with the spatial sections taken to be flat for simplicity, and time-dependent moduli fields

$$\phi_A = \phi_A(\tau) \,, \qquad z_i = z_i(\tau) \,. \tag{4.2}$$

Here, $\alpha(\tau)$ is the scale factor of the universe and $\nu(\tau)$ is the lapse function, which we will determine later. From eq. (3.6) the equations of motion for the blow-up moduli z_i are given by

$$\frac{d}{d\tau}\left(e^{-\mathbf{P}_i\cdot\boldsymbol{\phi}+3\alpha-\nu}\dot{z}_i\right) = 0 \,, \tag{4.3}$$

where the dot denotes the derivative with respect to τ. They can be easily integrated to

$$\dot{z}_i = \zeta_i \, e^{\mathbf{P}_i\cdot\boldsymbol{\phi}-3\alpha+\nu} \,, \tag{4.4}$$

with arbitrary integration constants ζ_i. Let us now choose the particularly convenient gauge $\nu = 3\alpha$. In this gauge, the sum of the (00) component and the spatial components of the Einstein equations reads $\ddot{\alpha} = 0$ and, we have

$$\alpha = v\tau + \alpha_0 \tag{4.5}$$

where v and α_0 are arbitrary integration constants. Proper time t is obtained by integrating

$$dt = e^{3\alpha} d\tau \tag{4.6}$$

which leads to

$$v\tau + \alpha_0 = \frac{1}{3}\ln\left(\sqrt{3E}|t-t_0|\right) . \tag{4.7}$$

Here t_0 is another integration constant and

$$E = 3v^2 \,. \tag{4.8}$$

JHEP01(2004)045

The solution (4.5) for the scale factor, written in proper time, then takes the form

$$\alpha = \frac{1}{3}\ln\left(\sqrt{3E}|t-t_0|\right) \tag{4.9}$$

and shows the expected power-law behaviour, with power 1/3, characteristic of evolution driven by kinetic energy. As usual, we have two branches, the negative-time branch, $t-t_0<0$, which ends in a future curvature singularity, and the positive-time branch, $t-t_0>0$, which starts out in a past curvature singularity.

We still have to consider the equations of motion for the bulk moduli ϕ_A. Replacing the blow-up moduli with eq. (4.4) and working in the gauge $\nu = 3\alpha$ they take the form

$$\frac{1}{2}\frac{d^2\phi_A}{d\tau^2} + \frac{\partial V}{\partial \phi_A} = 0\,. \tag{4.10}$$

The "effective" potential

$$V = 4\sum_i \zeta_i^2\, e^{\mathbf{P}_i\cdot\boldsymbol{\phi}} \tag{4.11}$$

in these equations of course originates from integrating out the blow-up moduli. The above equations of motion can be obtained from the lagrangian

$$\mathcal{L} = \frac{1}{4}\dot{\boldsymbol{\phi}}\cdot\dot{\boldsymbol{\phi}} - V\,. \tag{4.12}$$

They have to be supplemented by the hamiltonian constraint

$$\mathcal{H} = \frac{1}{4}\dot{\boldsymbol{\phi}}\cdot\dot{\boldsymbol{\phi}} + V = E\,, \tag{4.13}$$

which is simply the Friedmann equation rewritten in our language.

For the subsequent discussion it is useful to express the quantities measuring the geometry of the G_2 manifold in terms of the redefined fields. The volume $\mathcal{V}_0$ of the underlying orbifold, defined in eq. (2.6), can then be written as

$$\mathcal{V}_0 = \frac{1}{8}\exp\left(\frac{1}{3}\sum_A \phi_A\right), \tag{4.14}$$

where we have neglected higher order terms. Likewise, the fraction $\epsilon_{(\tau,n)}$ of the total volume taken up by the blow-up (τ,n) now takes the form

$$\epsilon_{(\tau,n)} = \frac{8}{3}\sum_a e^{-\mathbf{P}_i\cdot\boldsymbol{\phi}}\, z^2_{(\tau,n,a)}\,. \tag{4.15}$$

Let us summarise what we have achieved so far. We have integrated out the blow-up moduli and decoupled the scale factor by choosing a particular gauge. This has reduced the problem of analysing the evolution of 43 freely rolling G_2 moduli to one that only involves the seven bulk moduli subject to the potential (4.11). The remaining problem is, therefore, to solve the seven equations of motion (4.10) for the bulk moduli and the constraint (4.13). Each solution for the bulk moduli can then be inserted into eq. (4.4), which determines the corresponding evolution of the blow-up moduli.

JHEP01(2004)045

There exists a well-developed solution theory [21]–[26] for the lagrangian (4.12) and the associated equations of motion (4.10) for the bulk moduli if the system is of Toda type. To discuss this more specifically we introduce the matrix

$$A_{ij} = \mathbf{P}_i \cdot \mathbf{P}_j \,, \tag{4.16}$$

which consists of one or more irreducible blocks along the diagonal. The system is called Toda if each of these blocks is proportional to the Cartan matrix of a simple Lie-group. One can now inspect the vectors $\mathbf{P}_i = -\mu_i \mathbf{p}_i$ from table 1, where we recall that the μ_i are arbitrary signs. It is clear that the complete nine-dimensional matrix obtained from (4.16) is not proportional to a Cartan matrix for any choices of the signs μ_i and, hence, the system is not of Toda type in this case. However, note that the potential (4.11) does not necessarily contain all possible terms. In fact, each term is multiplied by an arbitrary integration constant ζ_i that can be set to zero. Such a vanishing ζ_i implies, from eq. (4.4), that the corresponding blow-up modulus z_i is constant. Hence, we learn that we can consistently freeze an arbitrary subset of blow-up moduli z_i and thereby select an arbitrary subset of the vectors $\mathbf{P}_i$ that appear in the potential (4.11). Then, for suitable subsets and choices of signs μ_i, the associated matrix A can well be proportional to a Cartan matrix, as we will see for several explicit examples studied further below. The system is then Toda and can be integrated analytically. The resulting solutions are, of course, special in that they rely on a number of blow-up moduli being frozen. However, in the present context, this seems to be the best one can do by analytic methods.

Before we go on to analyse explicit examples, it is worth making a general observation about the structure of the solutions. It is well-known [24, 25], for lagrangians of the type (3.6), that the fields z_i approach constant values asymptotically at early and late times (both in the negative and the positive time branch). This behaviour will also be confirmed in the explicit examples. Further note that the potential (4.11) consists of a sum of positive terms. Therefore, one expects the modes $\mathbf{P}_i \cdot \boldsymbol{\phi}$ to roll up the exponential slope at early time, then turn around and roll down at late time. In other words, the exponentials $\exp(\mathbf{P}_i \cdot \boldsymbol{\phi})$ decrease both in the past and in the future, at least if they explicitly appear in V, due to their associated integration constants ζ_i being non-zero. Hence, for the moduli that are not completely frozen, the associated relative volumes of the blow-ups, $\epsilon_{(\tau,n)}$, always increases asymptotically at early and late times. We will confirm this general feature explicitly in our examples.

5. A universal solution

As a warm-up, we would first like to consider a universal solution where all bulk moduli and all blow-up moduli evolve in the same way. It appears that a natural way to obtain such a solution is to start with an Ansatz where the seven bulk moduli are equal, that is where there is only a single breathing mode. However, it turns out that such an Ansatz is incompatible with the equations of motion (4.10). The reason behind this is that not all bulk moduli are on the same footing because of their differing couplings to the blow-up moduli, as is evident from the coupling vectors in table 1. Instead, we start with the slightly more general Ansatz

$$\phi_A = c_A \phi \tag{5.1}$$

where ϕ is the breathing mode and c_A are constants to be determined. For this Ansatz to be successful we need the seven equations of motion (4.10) for ϕ_A to reduce to a single equation for ϕ. This leads to the conditions

$$\mathbf{p}_{(\tau,a)} \cdot \mathbf{c} = \text{const} \tag{5.2}$$

$$\sum_{\tau,a} \zeta_{(\tau,a)}\, p_{(\tau,a)A} = \frac{1}{2} c_A \zeta^2 \tag{5.3}$$

where we have defined

$$\zeta^2_{(\tau,a)} = \sum_n \zeta^2_{(\tau,n,a)}\,, \tag{5.4}$$

and ζ is a constant. These conditions lead to a unique solution

$$\mathbf{c} = \frac{2}{7}(4,4,4,3,3,3,3) \tag{5.5}$$

for the vector $\mathbf{c}$ in our Ansatz (5.1). It can further be shown that all conditions (5.2), (5.3) can be satisfied for specific choices of coefficients $\zeta_{(\tau,a)}$ within a three-parameter family of solutions. The details of this are inessential for our subsequent discussion. For such a solution, the seven equations of motion (4.10) reduce to the single equation

$$\frac{1}{2}\ddot{\phi} - 2\mu\zeta^2 e^{-2\mu\phi} = 0\,, \tag{5.6}$$

where μ is an arbitrary sign. This equation can be easily integrated once and the integration constant can be fixed from the constraint (4.13). This leads to the first integral

$$E = |\mathbf{c}|^2 \left(\frac{1}{4}\dot{\phi}^2 + \zeta^2 e^{-2\mu\phi}\right) \tag{5.7}$$

and the solution

$$\phi = \mu \ln\cosh(y) + \frac{\mu}{2}\ln\left(\frac{|\mathbf{c}|^2\zeta^2}{E}\right) \tag{5.8}$$

for the breathing mode ϕ, where

$$y = \frac{2\sqrt{E}}{|\mathbf{c}|^2}(\tau - \tau_1) \tag{5.9}$$

is a rescaled time variable and τ_1 is a constant. The seven bulk moduli are proportional to ϕ and can be obtained by inserting this result into eq. (5.1). From eq. (4.4) one can then, in turn, obtain the solutions for the blow-up moduli. After another integration one finds

$$z_i = \frac{d_i}{2}\left(\tanh(y) + 1\right) + z_{0i} \tag{5.10}$$

where

$$d_i = \frac{\sqrt{E}\zeta_i}{\zeta^2} \tag{5.11}$$

JHEP01(2004)045

and z_{0i} are independent integration constants. As advertised earlier, the blow-up moduli z_i indeed approach constants asymptotically. More precisely, at early time, $y \to -\infty$, we have $z_i \to z_{0i}$ and at late time, $y \to \infty$, we have $z_i \to z_{1i} \equiv z_{0i} + d_i$, that is, d_i is the distance by which the blow-up modulus z_i moves. We also note that the non-trivial evolution of z_i happens around the time $y \simeq 0$.

Let us now interpret this solution. From eq. (4.14) the orbifold volume is directly measured by the breathing mode ϕ via

$$\mathcal{V}_0 = \frac{1}{8} \exp\left(\frac{16}{7}\phi\right) . \tag{5.12}$$

Further, from eq. (4.15), the relative volume of a blow-up is given by

$$\epsilon_i = \frac{8}{3} e^{2\mu\phi} z_i^2 = \frac{2}{3} |\mathbf{c}|^2 \frac{\zeta_i^2}{\zeta^2} \left(\frac{z_{0i}}{d_i} e^{-y} + \frac{z_{1i}}{d_i} e^{y}\right)^2 . \tag{5.13}$$

We recall that our solution can only be trusted as long as $\epsilon_i \ll 1$. Since, from eq. (5.3), the pre-factor on the r.h.s. of the above equation is of order one (for at least one i) we have to require the term in bracket be smaller than one. This leads to two cases, namely

- $\epsilon_i \ll 1$ if $|z_{0i}/d_i| \ll 1$ and $\ln|z_{0i}/d_i| \ll y \ll -1$. In this time range the breathing mode evolves as $\phi \simeq -\mu y + \text{const}$.
- $\epsilon_i \ll 1$ if $|z_{1i}/d_i| \ll 1$ and $1 \ll y \ll \ln|d_i/z_{1i}|$. In this time range the breathing mode evolves as $\phi \simeq \mu y + \text{const}$.

In both cases the breathing mode and, hence, the orbifold volume $\mathcal{V}_0$, can increase or decrease depending on the choice of the sign μ. However, independent of this choice, the relative blow-up volume ϵ_i will always leave the allowed range $\epsilon_i \ll 1$ when one of the limits of the given time ranges is approached. This means that after a finite proper time, both in the past and in the future, at least one of the blow-ups will take up a significant portion of the space and the approximation on which our Kähler potential (2.1) is based brakes down. In particular this means an evolution towards a state with small blow-ups at late time, $y \to \infty$ is not possible.

6. Potential with a single exponential

Let us now analyse the solutions more systematically starting from simple patterns of evolution of the blow-up moduli and moving to more complicated ones.

Certainly, the simplest possibility is to freeze all blow-up moduli by setting all $\zeta_i = 0$ in eq. (4.4). In this case, the effective potential (4.11) for the bulk moduli vanishes identically and the equations of motion (4.10) for ϕ_A can be easily integrated. In proper time t, related to τ by eq. (4.7), one easily finds as the general solution in this case

$$\phi_A = q_A \ln\left(\frac{|t - t_0|}{T}\right) + k_A \tag{6.1}$$

JHEP01(2004)045

where t_0, T and k_A are constants. The expansion powers q_A satisfy

$$|\mathbf{q}|^2 = \frac{4}{3}, \tag{6.2}$$

which follows from the constraint (4.13), and are otherwise arbitrary. These are simply solutions describing power-law evolution of the bulk moduli, which, in fact, are identical to the ones that can be obtained for M-theory on a seven-dimensional torus.

We now move to the next more complicated case where blow-up moduli of only one particular type (τ, a) evolve non-trivially and all the others have been set to constants. We write $\mathbf{P} = \mathbf{P}_{(\tau,a)}$ and $\zeta^2 = 4\sum_n \zeta_{(\tau,n,a)}$ for simplicity of notation. The effective potential consists of only one term and takes the form

$$V = \zeta^2 e^{\mathbf{P}\cdot\boldsymbol{\phi}}. \tag{6.3}$$

This situation corresponds to an SU(2) Toda model so the general solution to eq. (4.10) can be found. It is given by

$$\boldsymbol{\phi} = \mathbf{p}^{(i)} \ln(x) + \left(\mathbf{p}^{(f)} - \mathbf{p}^{(i)}\right) \ln\left(1 + x^{\delta}\right)^{1/\delta} + \mathbf{k} \tag{6.4}$$

subject to the constraints

$$|\mathbf{p}^{(i)}|^2 = \frac{4}{3} \tag{6.5}$$

$$\delta = \mathbf{P}\cdot\mathbf{p}^{(i)} \tag{6.6}$$

$$\mathbf{p}^{(f)} = \mathbf{p}^{(i)} - \frac{2\mathbf{P}\cdot\mathbf{p}^{(i)}}{|\mathbf{P}|^2}\mathbf{P} \tag{6.7}$$

$$\exp(\mathbf{P}\cdot\mathbf{k}) = \frac{3E\delta^2}{|\mathbf{P}|^2\zeta^2}. \tag{6.8}$$

Further,

$$x = \frac{|t - t_0|}{T} \tag{6.9}$$

is the rescaled proper time and t_0 and T are constants. Note that eqs. (6.5) and (6.7) imply that

$$|\mathbf{p}^{(f)}|^2 = \frac{4}{3}. \tag{6.10}$$

This solution is invariant under the exchange of $\mathbf{p}^{(i)}$ and $\mathbf{p}^{(f)}$ and we remove this ambiguity by requiring $\delta \geq 0$ in the positive-time branch and $\delta \leq 0$ in the negative-time branch. The interpretation of these solutions is well-known [24, 25]. They interpolate between two, generally different "free" solutions (6.1), one with $\mathbf{q} = \mathbf{p}^{(i)}$ at early time and one with $\mathbf{q} = \mathbf{p}^{(f)}$ at late time. In these asymptotic regions the blow-up moduli $z_{(\tau,n,a)}$ are constant while they move around the time $x \simeq 1$ to facilitate the transition between the two free solutions.

One can now compute the relative volume $\epsilon_{(\tau,a)}$ of the blow-ups by inserting the solution (6.4) into eq. (4.15). As before, we find that the required condition $\epsilon_{(\tau,a)} \ll 1$ is satisfied only in two finite time windows at $x \ll 1$ and $x \gg 1$. At the endpoints of these windows the volume of the blow-ups becomes sizeable and control over the approximation is lost.

7. More complicated cases

An obvious generalisation of the previous example is to consider a situation corresponding to an $\mathrm{SU}(2)^n$ Toda model, for some integer n. This amounts to having a subset of blow-up moduli evolve non-trivially for which the associated characteristic vectors $\mathbf{P}_i$ are orthogonal, that is,

$$\mathbf{P}_i \cdot \mathbf{P}_j = 2\delta_{ij} \,. \tag{7.1}$$

Inspection of table 1 shows such cases can indeed be realised. For example, allowing the moduli of a certain type τ to evolve non-trivially, while moduli of the other two types are being kept constant by setting $\zeta_{\tau',n,a} = 0$ in eq. (4.4) for $\tau' \neq \tau$, leads to an $\mathrm{SU}(2)^3$ Toda model with three exponentials appearing in the potential (4.11). From table 1 there are a number of other options and in the following we will simply assume the existence of n vectors $\mathbf{P}_i$ satisfying (7.1) to cover all possibilities.

The standard procedure to deal with such a system is to introduce a new constant basis $\mathbf{e}_A = (\mathbf{e}_i, \mathbf{e}_a)$ in field space, where $i = 1, \cdots, n$ and $a = n+1, \cdots, 7$ such that

$$\mathbf{e}_i = \frac{1}{\sqrt{2}} \mathbf{P}_i \tag{7.2}$$

and the remaining vectors $\mathbf{e}_a$ are chosen to complete the set to an orthonormal basis satisfying

$$\mathbf{e}_A \cdot \mathbf{e}_B = \delta_{AB} \,. \tag{7.3}$$

The bulk fields $\boldsymbol{\phi}$ can then be expanded as

$$\boldsymbol{\phi} = \sum_A \rho_A \mathbf{e}_A \,, \tag{7.4}$$

where ρ_A is a new set of fields. The lagrangian (4.12) and the hamiltonian (4.13), written in terms of these new fields, take the form

$$\mathcal{L} = \frac{1}{4} \sum_A \dot{\rho}_A^2 - V \,, \qquad \mathcal{H} = \frac{1}{4} \sum_A \dot{\rho}_A^2 + V = E \,, \tag{7.5}$$

with the potential

$$V = 4 \sum_i \zeta_i^2 e^{2\rho_i} \,. \tag{7.6}$$

Note that the modes ρ_a have decoupled from this potential, which is, of course, one of the motivations behind introducing the basis $\mathbf{e}_A$.

The general solution to the lagrangian in (7.5) can be easily obtained as

$$\boldsymbol{\phi} = \sum_A \rho_A \mathbf{e}_A \tag{7.7}$$

with

$$\rho_a = k_a \tau + \tau_a \tag{7.8}$$

$$\rho_i = \sqrt{2}\ln\cosh(y_i) - \frac{1}{\sqrt{2}}\ln\left(\frac{16\zeta_i^2}{k_i^2}\right), \tag{7.9}$$

where τ_A and k_A are constants and

$$y_i = \frac{k_i}{\sqrt{2}}(\tau - \tau_i) \tag{7.10}$$

are rescaled time coordinates. The hamiltonian constraint in (7.5) amounts to the condition

$$\sum_A k_A^2 = 4E\,. \tag{7.11}$$

As expected the modes ρ_a evolve freely. Each of the other modes ρ_i evolves similarly to what has been found for the single SU(2) Toda model in the previous section. This can be seen explicitly by converting the above solution to proper time using (4.7) and compering with eq. (6.4). This means that each mode interpolates between two regions of simple power-law evolution at early and late time. The blow-up moduli z_i are constant in these asymptotic regions and their evolution at intermediate times facilitates the transition. This can be seen explicitly from their solution

$$z_i = \frac{d_i}{2}\left(\tanh(y_i) + 1\right) + z_{0i}\,, \tag{7.12}$$

with z_{0i} and d_i constants representing the value of the modulus at early time and its total change respectively. This solution is obtained by inserting the above solution for $\boldsymbol{\phi}$ into eq. (4.4).

From eq. (4.15) one finds the relative volume

$$\epsilon_i = \frac{\sqrt{2}}{3}\left(\frac{z_{1i}}{d_i}e^{y_i} + \frac{z_{0i}}{d_i}e^{-y_i}\right)^2 \tag{7.13}$$

of the blow-ups. As in all previous cases, ϵ_i can only be kept small, $\epsilon_i \ll 1$, for a finite proper time at either $y_i \ll -1$ or $y_i \gg 1$. For a valid solution, all ϵ_i need to be small, which amounts to arranging an overlap between those regions by choosing the time shifts τ_i appropriately. Outside this overlap region control of the approximation is lost as one or more blow-ups become large and take up a significant part of the internal space.

There are also cases leading to a Toda model associated with a higher-rank simple group. Consider, for example, the three vectors

$$\mathbf{P}_{(\alpha,1)} = (-1,0,0,0,0,-1,0) \tag{7.14}$$
$$\mathbf{P}_{(\beta,1)} = (1,0,0,0,0,0,1) \tag{7.15}$$
$$\mathbf{P}_{(\gamma,3)} = (0,-1,0,0,0,0,-1)\,, \tag{7.16}$$

obtained from table 1 with the sign choice $\mu_{(\alpha,1)} = -1$, $\mu_{(\beta,2)} = 1$ and $\mu_{(\gamma,3)} = -1$ and set all blow-up moduli with (τ, a) different from the above to constants. The matrix A in eq. (4.16) is then given by

$$(A_{\tau\sigma}) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \tag{7.17}$$

which is precisely the Cartan matrix of SU(4). Hence, we are dealing with an SU(4) Toda model.

As before, the first step in solving this model is to introduce a new basis $(\mathbf{e}_A) = (\mathbf{e}_\tau, \mathbf{e}_a)$, where $\tau = 1, 2, 3$ and $a = 4, \ldots, 7$, in field space. Here the three vectors $\mathbf{e}_\tau$ are identified with (7.14)–(7.16) and the remaining four vectors $\mathbf{e}_a$ are chosen to be orthonormal among themselves and orthogonal to all $\mathbf{e}_\tau$. Consequently, we have a basis satisfying

$$\mathbf{e}_\tau \cdot \mathbf{e}_\sigma = A_{\tau\sigma} , \qquad \mathbf{e}_\tau \cdot \mathbf{e}_a = 0 , \qquad \mathbf{e}_a \cdot \mathbf{e}_b = \delta_{ab} . \tag{7.18}$$

Expanding the fields $\boldsymbol{\phi}$

$$\boldsymbol{\phi} = \sum_A \rho_A \mathbf{e}_A \tag{7.19}$$

as before one finds for the lagrangian (4.12)

$$\mathcal{L} = \frac{1}{4} \sum_{\tau,\sigma} A_{\tau,\sigma} \dot{\rho}_\tau \dot{\rho}_\sigma + \frac{1}{4} \sum_a \dot{\rho}_a^2 - V \tag{7.20}$$

with potential

$$V = 4 \sum_\tau \zeta_\tau^2 \exp \left(\sum_\sigma A_{\tau\sigma} \rho_\sigma \right) . \tag{7.21}$$

The modes ρ_a are decoupled from this potential and their equations of motion immediately lead to the general solution

$$\rho_a = k_a (\tau - \tau_a) , \tag{7.22}$$

where k_a and τ_a are constants. Following the methods of ref. [21], the solutions for the other modes are obtained as

$$e^{-\rho_\tau} = \sum_{\boldsymbol{\lambda} \in \Lambda_\tau} b_\tau(\boldsymbol{\lambda}) \exp(\boldsymbol{\lambda} \cdot (\mathbf{k}\tau - \boldsymbol{\tau})) , \tag{7.23}$$

where $\mathbf{k}$ and $\boldsymbol{\tau}$ are constant vectors. The three sets Λ_τ contain the weights of the fundamental representation $\mathbf{4}$, the vector representation $\mathbf{6}$ and the anti-fundamental representation $\bar{\mathbf{4}}$ of SU(4), respectively, and are explicitly given by

$$\Lambda_1 = \{(100), (-110), (0-11), (00-1)\} \tag{7.24}$$
$$\Lambda_2 = \{(010), (1-11), (-101), (10-1), (-11-1), (0-10)\} \tag{7.25}$$
$$\Lambda_3 = \{(001), (01-1), (1-10), (-100)\} . \tag{7.26}$$

Finally, $b_\tau(\boldsymbol{\lambda})$ represents a set of constants which depends on ζ_τ as well as on $\mathbf{k}$. They can be calculated by inserting the solutions (7.23) into the equations of motion but their explicit form will not be of any relevance here.

The important feature of (7.23) is that asymptotically one of the exponentials in the sum will dominate leading to a power-law evolution. In these regions, the blow-up moduli are approximately constant and the relative blow-up volumes increase until they approach values of order one where the approximation underlying our analysis breaks down. Hence, this more complicated Toda solution also conforms with our general expectation of asymptotically increasing blow-up volumes.

JHEP01(2004)045

Acknowledgments

A. L. is supported by a PPARC Advanced Fellowship, and S. M. by a PPARC Postgraduate Studentship.

References

[1] G. Papadopoulos and P.K. Townsend, *Compactification of $D = 11$ supergravity on spaces of exceptional holonomy*, *Phys. Lett.* **B 357** (1995) 300 [hep-th/9506150].

[2] B.S. Acharya, *M-theory, Joyce orbifolds and super Yang-Mills*, *Adv. Theor. Math. Phys.* **3** (1999) 227 [hep-th/9812205].

[3] S. Gukov, *Solitons, superpotentials and calibrations*, *Nucl. Phys.* **B 574** (2000) 169 [hep-th/9911011].

[4] M. Atiyah and E. Witten, *M-theory dynamics on a manifold of G_2 holonomy*, *Adv. Theor. Math. Phys.* **6** (2003) 1 [hep-th/0107177].

[5] B.S. Acharya and B. Spence, *Flux, supersymmetry and M-theory on 7-manifolds*, hep-th/0007213.

[6] M. Cvetič, G. Shiu and A.M. Uranga, *Chiral four-dimensional $N = 1$ supersymmetric type-IIA orientifolds from intersecting d6-branes*, *Nucl. Phys.* **B 615** (2001) 3 [hep-th/0107166].

[7] E. Witten, *Anomaly cancellation on G_2 manifolds*, hep-th/0108165.

[8] B. Acharya and E. Witten, *Chiral fermions from manifolds of G_2 holonomy*, hep-th/0109152.

[9] E. Witten, *Deconstruction, G_2 holonomy and doublet-triplet splitting*, in *Hamburg 2002, supersymmetry and unification of fundamental interactions*, vol. 1 pp. 472–491 [hep-ph/0201018].

[10] C. Beasley and E. Witten, *A note on fluxes and superpotentials in M-theory compactifications on manifolds of G_2 holonomy*, *J. High Energy Phys.* **07** (2002) 046 [hep-th/0203061].

[11] C.I. Lazaroiu and L. Anguelova, *M-theory compactifications on certain 'toric' cones of G_2 holonomy*, *J. High Energy Phys.* **01** (2003) 066 [hep-th/0204249].

[12] L. Anguelova and C.I. Lazaroiu, *M-theory on 'toric' G_2 cones and its type-II reduction*, *J. High Energy Phys.* **10** (2002) 038 [hep-th/0205070].

[13] P. Berglund and A. Brandhuber, *Matter from G_2 manifolds*, *Nucl. Phys.* **B 641** (2002) 351 [hep-th/0205184].

[14] K. Behrndt, G. Dall'Agata, D. Lüst and S. Mahapatra, *Intersecting 6-branes from new 7-manifolds with G_2 holonomy*, *J. High Energy Phys.* **08** (2002) 027 [hep-th/0207117].

[15] T. Friedmann and E. Witten, *Unification scale, proton decay and manifolds of G_2 holonomy*, *Adv. Theor. Math. Phys.* **7** (2003) 577 [hep-th/0211269].

[16] B.S. Acharya, *A moduli fixing mechanism in M-theory*, hep-th/0212294.

[17] A. Lukas and S. Morris, *Moduli Kähler potential for M-theory on a G_2 manifold*, hep-th/0305078, to appear in *Phys. Rev.* **D**.

[18] D. Joyce, *Compact riemannian 7-Manifolds with Holonomy G_2, I*, *J. Diff. Geom.* **43** (1996) 291.

[19] D. Joyce, *Compact riemannian 7-Manifolds with Holonomy G_2, II*, *J. Diff. Geom.* **43** (1996) 329.

[20] D. Joyce, *Compact manifolds with special holonomy*, Oxford University Press, Oxford 2000.

[21] B. Kostant, *The solution to a generalized toda lattice and representation theory*, *Adv. Math.* **34** (1979) 195.

[22] H. Lu, C.N. Pope and K.W. Xu, *Liouville and toda solutions of M-theory*, *Mod. Phys. Lett.* **A 11** (1996) 1785 [hep-th/9604058].

[23] H. Lu and C.N. Pope, SL$(N+1,R)$ *toda solitons in supergravities*, *Int. J. Mod. Phys.* **A 12** (1997) 2061 [hep-th/9607027].

[24] A. Lukas, B.A. Ovrut and D. Waldram, *Cosmological solutions of type-II string theory*, *Phys. Lett.* **B 393** (1997) 65 [hep-th/9608195].

[25] A. Lukas, B.A. Ovrut and D. Waldram, *String and M-theory cosmological solutions with ramond forms*, *Nucl. Phys.* **B 495** (1997) 365 [hep-th/9610238].

[26] A. Lukas, B.A. Ovrut and D. Waldram, *Stabilizing dilaton and moduli vacua in string and M-theory cosmology*, *Nucl. Phys.* **B 509** (1998) 169 [hep-th/9611204].

JHEP01(2004)045